Synopsis for Ph.D. Programme

Title of Research:
A study on some generalized normed linear spaces and fixed point theory in such spaces

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Introduction

The formal definition of norm function on a vector space was given by Banach [1] in 1922. This formulation of the axiomatic definition of norm brings the opportunity to the mathematicians to generalize the notion of norm to meet the necessity of different theoritical and physical needs. In this direction, first significant contribution was made by Vulich [27]. He introduced the idea of K-normed space in 1937. However this generalized notion did not catch much attention of the functional analysis community.

German mathematician Gähler introduced 2-metric in 1963 as a notion of distance between three points of an arbitrary set [10]. As a consequence he defined 2-norm in 1964 [11]. The definition of 2-metric and 2-norm were motivated from the idea of area of a triangle. Gähler continued to study geometry of 2-normed spaces [12]. He also introduced n-metric and n-norm as further generalization of his work in his subsequent articles [13–15]. White [28] introduced 2-Banach space in 1969. Later Diminnie, Gähler and White jointly introduced and studied 2-inner product space [5, 6]. As further generalization, Misiak introduced n-inner product space [22] in 1989. Strictly convex and strictly 2-convex 2-normed spaces were introduced by Diminnie, Gähler and White [7] in 1974. Further studies on convexity were done by various research groups lead by Ha, Cho, Freese and Kim (For reference please see [9, 17, 18, 20, 21]). Gunawan and Mashadi studied the convergence of sequences in 2-normed spaces and derived a norm from a given 2-norm in such a way that convergence of sequence for both the spaces are equivalent [16]. They extended their result for n-normed spaces in [16]. Recently in 2015, a new kind of continuity has been introdued in 2-normed spaces by Ersan and Çakalli in [8]

In another process of generalization, B. C. Dhage defined D-metric space in 1992 as a generalization of metric space [3]. He also studied some topological properties of this space in [4]. But the topological structure of D-metric space turned out to be defective [23, 25]. To get over this situation, Mustafa and Sims modified the definition of D-metric and introduced G-metric in 2006 [24]. Sedghi, Shobe and Aliouche modified D-metric in another way and defined S-metric in 2012 [26]. Following the idea of G-metric, K. A. Khan defined and studied G-norm in [19]. On the other hand, Parin Chaipunya and Poom Kumam developed a notion to bring all the existing ideas of distance between three points under the same umbrella by defining *g-3ps* space [2].

Proposed plan of work

Following the history of research works done towards the generalization of metric and normed linear spaces, we can say that researchers are trying to find a proper generalization in various ways. 2-metric and 2-norm are some old developments whereas G-metric, and G-norm are new. Searching for some proper generalization is still a continuing activity. My concentration will be on the generalization of normed linear space. Specifically, I have a plan to study the following topics during my Ph.D. work:

- To study various possible ways to generalize the idea of normed linear spaces.
- To study the topological structure of those spaces.

- To develop functional analysis in those generalized normed linear spaces. Specifically we want to study the analogous results of four fundamental theorems, viz., open mapping theorem, closed graph theorem, Hahn-Banach theorem and uniform boundedness theorem.
- To study the geometric structure of vector space with respect to the generalized norms. Especially we are interested in convexity, uniform convexity and orthogonality properties.
- To study fixed point theory in those spaces.

Following to the above plan, I have already studied 2-normed space, G-metric space, G-normed space and *g-3ps* space and developed some results. So far we have been able to characterize topological structure of 2-normed space and G-normed space. We have done the following two works:

(i) The topology of G-normed space is studied and it is shown that the space is always normable. So topologically there is nothing new in this space. Further, a new notion of generalized norm in a vector space, called General 2-Norm, is introduced and its topological structure is studied.

(ii) A separating family of seminorms is induced in a 2-normed space and the problems of metrizability and normability are investigated in such space.

We have written the following research papers based on those developments:

1. A New Generalization of Normed Linear Space (Communicated to "Topology and its Applications").
2. On metrizability and normability of 2-normed spaces (Communicated to "Mathematical Sciences").

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